Convergence and Exponential Ergodicity of Regime-Switching Stochastic Functional Differential Equations with Infinite Delay

Fubao Xi
Beijing Institute of Technology
xifb@bit.edu.cn

(Joint work with Jun Li)

The 18th International Workshop on Markov Processes and Related Topics
July 30th-August 2nd, 2023
Tianjin University

Outline

Switching Models and Dynamics with Delay

In order to model random environment or other random factors, a lot of **switching models** such as switching diffusion processes and switching jump-diffusion processes have been studied extensively.

More often than not, delay (or memory) is ubiquitous and inevitable in the real world. To deal with more realistic situation, efforts have also been devoted to the development of dynamic models that take the influence of past history into consideration.

Hence, dynamics with **delay (or memory)** have also been studied extensively.

Today, we would like to consider a class of **regime-switching diffusion processes with infinite delay**.

Switching Models and Dynamics with Delay

In order to model random environment or other random factors, a lot of **switching models** such as switching diffusion processes and switching jump-diffusion processes have been studied extensively.

More often than not, delay (or memory) is ubiquitous and inevitable in the real world. To deal with more realistic situation, efforts have also been devoted to the development of dynamic models that take the influence of past history into consideration.

Hence, dynamics with **delay (or memory)** have also been studied extensively.

Today, we would like to consider a class of **regime-switching diffusion processes with infinite delay**.

Switching Models and Dynamics with Delay

In order to model random environment or other random factors, a lot of **switching models** such as switching diffusion processes and switching jump-diffusion processes have been studied extensively.

More often than not, delay (or memory) is ubiquitous and inevitable in the real world. To deal with more realistic situation, efforts have also been devoted to the development of dynamic models that take the influence of past history into consideration.

Hence, dynamics with **delay (or memory)** have also been studied extensively.

Today, we would like to consider a class of **regime-switching diffusion processes with infinite delay**.

Switching Models and Dynamics with Delay

In order to model random environment or other random factors, a lot of **switching models** such as switching diffusion processes and switching jump-diffusion processes have been studied extensively.

More often than not, delay (or memory) is ubiquitous and inevitable in the real world. To deal with more realistic situation, efforts have also been devoted to the development of dynamic models that take the influence of past history into consideration.

Hence, dynamics with **delay (or memory)** have also been studied extensively.

Today, we would like to consider a class of **regime-switching diffusion processes with infinite delay**.

Model Construction

Let $C((-\infty,0];\mathbb{R}^d)$ be the family of continuous functions from $(-\infty,0]$ to $\mathbb{R}^d$.

Given a real number $r>0$ and an integer $N\geq 1$, **set**

$$C_r := \left\{\varphi \in C((-\infty,0];\mathbb{R}^d);\ \|\varphi\|_r := \sup_{-\infty<\theta\leq 0} e^{r\theta}|\varphi(\theta)| < \infty\right\},$$

$$\mathbb{S} := \{1,2,\ldots,N\}.$$

The bigger the positive number r, the more the space C_r contains.

Model Construction

Let $C((-\infty,0];\mathbb{R}^d)$ be the family of continuous functions from $(-\infty,0]$ to $\mathbb{R}^d$.

Given a real number $r>0$ and an integer $N\geq 1$, **set**

$$C_r := \Big\{\varphi\in C((-\infty,0];\mathbb{R}^d);\ \|\varphi\|_r := \sup_{-\infty<\theta\leq 0} e^{r\theta}|\varphi(\theta)|<\infty\Big\},$$

$$\mathbb{S} := \{1,2,\ldots,N\}.$$

The bigger the positive number r, the more the space C_r contains.

Model Construction

Let $C((-\infty,0];\mathbb{R}^d)$ be the family of continuous functions from $(-\infty,0]$ to $\mathbb{R}^d$.

Given a real number $r>0$ and an integer $N\geq 1$, **set**

$$C_r := \left\{\varphi\in C((-\infty,0];\mathbb{R}^d);\ \|\varphi\|_r := \sup_{-\infty<\theta\leq 0} e^{r\theta}|\varphi(\theta)|<\infty\right\},$$

$$\mathbb{S} := \{1,2,\ldots,N\}.$$

The bigger the positive number r, the more the space C_r contains.

Model Construction (cont'd)

Let $X(t)$ satisfy the following stochastic functional differential equation with infinite delay

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(X_t, \Lambda(t))\mathrm{d}W(t), \quad t \geq 0 \tag{1}$$

with initial data $X_0 = \varphi \in C_r$ and $\Lambda(0) = i \in \mathbb{S}$, where

$$X_t(\theta) := X(t+\theta), \quad -\infty < \theta \leq 0,$$

is the so-called segment process of $X(t)$ (i.e., solution map), and $W(t)$ is an d-dimensional Brownian motion. Let $\Lambda(t)$ be a jump process on the state space $\mathbb{S}$ with the transition kernel such that for any $k, l \in \mathbb{S}$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } \ l \neq k, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } \ l = k, \end{cases} \tag{2}$$

provided $\Delta \downarrow 0$.

Model Construction (cont'd)

Let $X(t)$ satisfy the following stochastic functional differential equation with infinite delay

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(X_t, \Lambda(t))\mathrm{d}W(t), \quad t \geq 0 \tag{1}$$

with initial data $X_0 = \varphi \in C_r$ and $\Lambda(0) = i \in \mathbb{S}$, where

$$X_t(\theta) := X(t+\theta), \quad -\infty < \theta \leq 0,$$

is the so-called segment process of $X(t)$ (i.e., solution map), and $W(t)$ is an d-dimensional Brownian motion. Let $\Lambda(t)$ be a jump process on the state space $\mathbb{S}$ with the transition kernel such that for any $k, l \in \mathbb{S}$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } l \neq k, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } l = k, \end{cases} \tag{2}$$

provided $\Delta \downarrow 0$.

Model Construction (cont'd)

Let $X(t)$ satisfy the following stochastic functional differential equation with infinite delay

$$\mathrm{d}X(t) = b(X_t, \Lambda(t))\mathrm{d}t + \sigma(X_t, \Lambda(t))\mathrm{d}W(t), \quad t \geq 0 \tag{1}$$

with initial data $X_0 = \varphi \in C_r$ and $\Lambda(0) = i \in \mathbb{S}$, where

$$X_t(\theta) := X(t+\theta), \quad -\infty < \theta \leq 0,$$

is the so-called segment process of $X(t)$ (i.e., solution map), and $W(t)$ is an d-dimensional Brownian motion. Let $\Lambda(t)$ be a jump process on the state space $\mathbb{S}$ with the transition kernel such that for any $k, l \in \mathbb{S}$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases} q_{kl}(x)\Delta + o(\Delta), & \text{if } \ l \neq k, \\ 1 + q_{kk}(x)\Delta + o(\Delta), & \text{if } \ l = k, \end{cases} \tag{2}$$

provided $\Delta \downarrow 0$.

A "Sample Path" of a Switching Diffusion with Delay

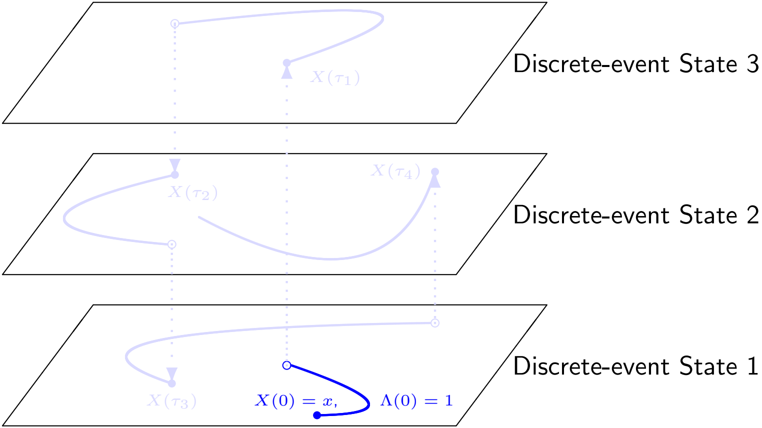

Figure: A "Sample Path" of a Switching Diffusion with Delay

A “Sample Path” of a Switching Diffusion with Delay

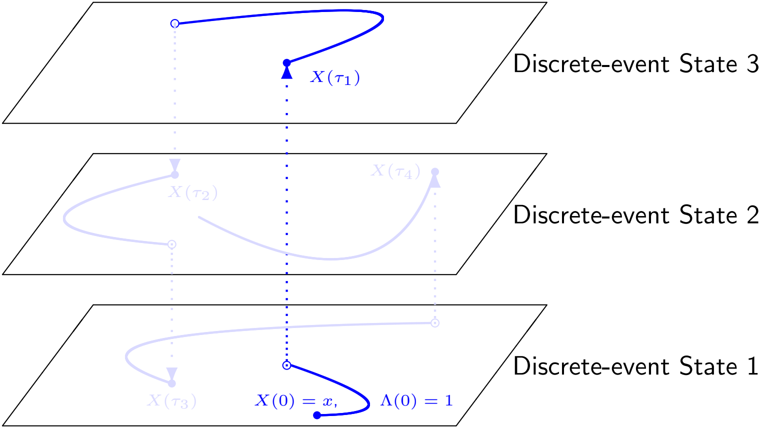

Figure: A “Sample Path” of a Switching Diffusion with Delay

A "Sample Path" of a Switching Diffusion with Delay

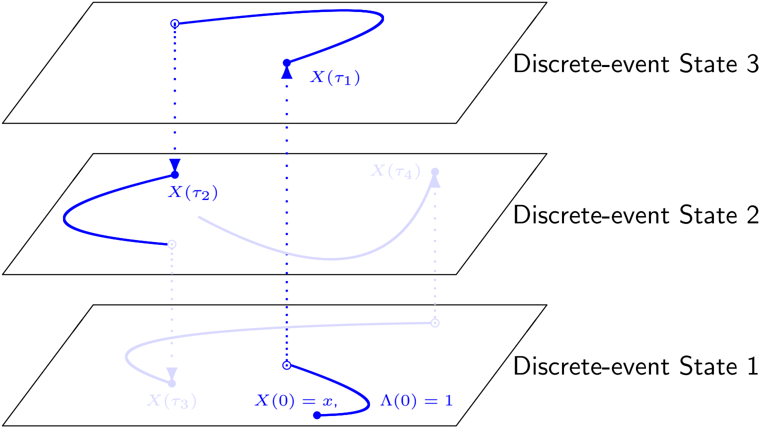

Figure: A "Sample Path" of a Switching Diffusion with Delay

A "Sample Path" of a Switching Diffusion with Delay

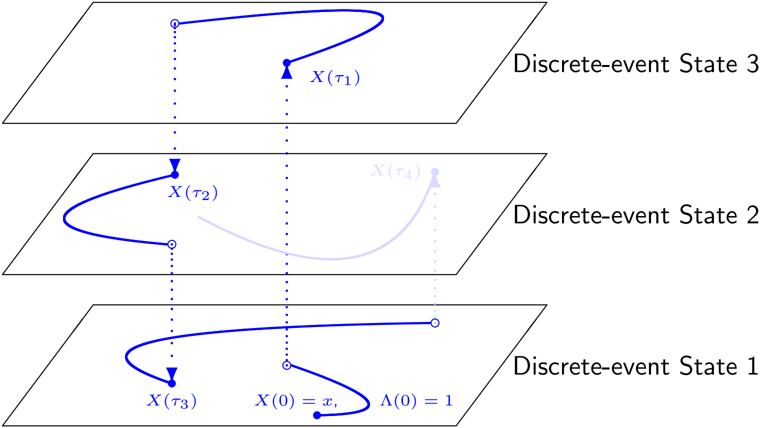

Figure: A "Sample Path" of a Switching Diffusion with Delay

A “Sample Path” of a Switching Diffusion with Delay

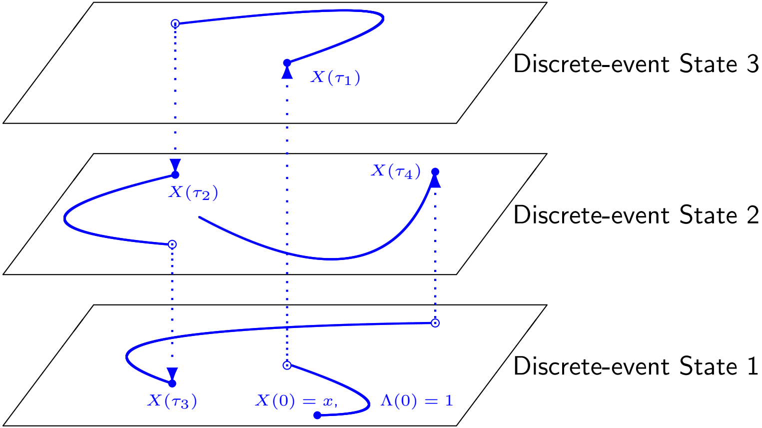

Figure: A “Sample Path” of a Switching Diffusion with Delay

A "Sample Path" of a Switching Diffusion with Delay

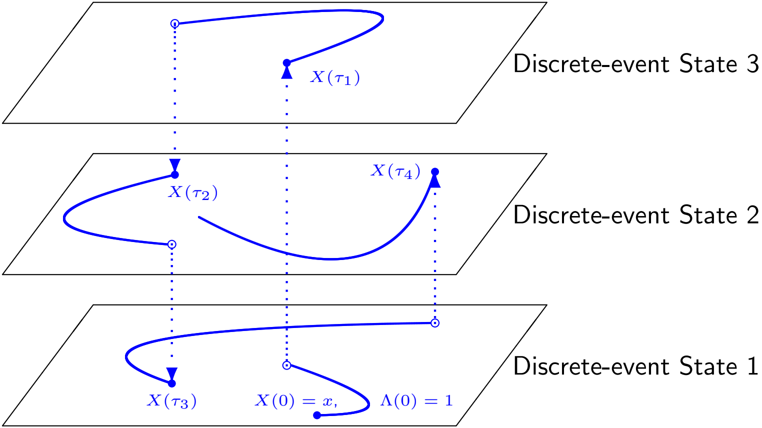

Figure: A "Sample Path" of a Switching Diffusion with Delay

Some Notation

Write $\mathbb{R}_- = (-\infty, 0]$. Let $\mathcal{P}(\mathbb{R}_-)$ denote the set of probability measures on $\mathbb{R}_-$. Moreover, for any given $c > 0$, define

$$\mathcal{P}_c(\mathbb{R}_-) := \left\{ \rho \in \mathcal{P}(\mathbb{R}_-); \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) < \infty \right\}.$$

Lemma 1 (Wu, Yin and Mei (2017))

Fix $c_0 > 0$ *and* $\rho \in \mathcal{P}_{c_0}(\mathbb{R}_-)$. *For any* $0 < c < c_0$,

$$\int_{-\infty}^{0} e^{-c_0\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} \rho(\mathrm{d}\theta) = 1,$$

$$\mathcal{P}_{c_0}(\mathbb{R}_-) \subset \mathcal{P}_c(\mathbb{R}_-) \subset \mathcal{P}(\mathbb{R}_-).$$

The bigger the positive constant c, *the less the set* $\mathcal{P}_c(\mathbb{R}_-)$ *contains.*

Some Notation

Write $\mathbb{R}_- = (-\infty, 0]$. Let $\mathcal{P}(\mathbb{R}_-)$ denote the set of probability measures on $\mathbb{R}_-$. Moreover, for any given $c > 0$, define

$$\mathcal{P}_c(\mathbb{R}_-) := \left\{ \rho \in \mathcal{P}(\mathbb{R}_-); \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) < \infty \right\}.$$

Lemma 1 (Wu, Yin and Mei (2017))

Fix $c_0 > 0$ and $\rho \in \mathcal{P}_{c_0}(\mathbb{R}_-)$. For any $0 < c < c_0$,

$$\int_{-\infty}^{0} e^{-c_0\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} \rho(\mathrm{d}\theta) = 1,$$

$$\mathcal{P}_{c_0}(\mathbb{R}_-) \subset \mathcal{P}_c(\mathbb{R}_-) \subset \mathcal{P}(\mathbb{R}_-).$$

The bigger the positive constant c, the less the set $\mathcal{P}_c(\mathbb{R}_-)$ contains.

Some Notation

Write $\mathbb{R}_- = (-\infty, 0]$. Let $\mathcal{P}(\mathbb{R}_-)$ denote the set of probability measures on $\mathbb{R}_-$. Moreover, for any given $c > 0$, define

$$\mathcal{P}_c(\mathbb{R}_-) := \left\{ \rho \in \mathcal{P}(\mathbb{R}_-); \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) < \infty \right\}.$$

Lemma 1 (Wu, Yin and Mei (2017))

Fix $c_0 > 0$ and $\rho \in \mathcal{P}_{c_0}(\mathbb{R}_-)$. For any $0 < c < c_0$,

$$\int_{-\infty}^{0} e^{-c_0\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} \rho(\mathrm{d}\theta) = 1,$$

$$\mathcal{P}_{c_0}(\mathbb{R}_-) \subset \mathcal{P}_c(\mathbb{R}_-) \subset \mathcal{P}(\mathbb{R}_-).$$

The bigger the positive constant c, the less the set $\mathcal{P}_c(\mathbb{R}_-)$ contains.

Some Notation

Write $\mathbb{R}_- = (-\infty, 0]$. Let $\mathcal{P}(\mathbb{R}_-)$ denote the set of probability measures on $\mathbb{R}_-$. Moreover, for any given $c > 0$, define

$$\mathcal{P}_c(\mathbb{R}_-) := \left\{ \rho \in \mathcal{P}(\mathbb{R}_-); \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) < \infty \right\}.$$

Lemma 1 (Wu, Yin and Mei (2017))

Fix $c_0 > 0$ *and* $\rho \in \mathcal{P}_{c_0}(\mathbb{R}_-)$. *For any* $0 < c < c_0$,

$$\int_{-\infty}^{0} e^{-c_0\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} e^{-c\theta} \rho(\mathrm{d}\theta) > \int_{-\infty}^{0} \rho(\mathrm{d}\theta) = 1,$$

$$\mathcal{P}_{c_0}(\mathbb{R}_-) \subset \mathcal{P}_c(\mathbb{R}_-) \subset \mathcal{P}(\mathbb{R}_-).$$

The bigger the positive constant c, *the less the set* $\mathcal{P}_c(\mathbb{R}_-)$ *contains.*

Hypotheses

Assume $b : C_r \times \mathbb{S} \to \mathbb{R}^n$ and $\sigma : C_r \times \mathbb{S} \to \mathbb{R}^n \otimes \mathbb{R}^n$ satisfy the following conditions:

(A1) For any $k \in \mathbb{S}$ and any $M > 0$, there exists a constant $H_M > 0$ such that

$$|b(\varphi, k) - b(\psi, k)| \vee |\sigma(\varphi, k) - \sigma(\psi, k)| \leq H_M \|\varphi - \psi\|_r$$

for those φ, $\varphi \in C_r$ with $\|\varphi\|_r \vee \|\psi\|_r \leq M$.

(A2) For all φ, $\psi \in C_r$ and $k \in \mathbb{S}$, there exist $\alpha(k) \in \mathbb{R}$, $\beta(k) \in \mathbb{R}_+$ and $\rho \in \mathcal{P}_{2r}(\mathbb{R}_-)$ such that

$$\begin{aligned} &2\langle \varphi(0) - \psi(0), b(\varphi, k) - b(\psi, k)\rangle \\ &\quad \leq \alpha(k)|\varphi(0) - \psi(0)|^2 + \beta(k) \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta). \end{aligned}$$

Hypotheses

Assume $b : C_r \times \mathbb{S} \to \mathbb{R}^n$ and $\sigma : C_r \times \mathbb{S} \to \mathbb{R}^n \otimes \mathbb{R}^n$ satisfy the following conditions:

(A1) For any $k \in \mathbb{S}$ and any $M > 0$, there exists a constant $H_M > 0$ such that

$$|b(\varphi, k) - b(\psi, k)| \vee |\sigma(\varphi, k) - \sigma(\psi, k)| \le H_M \|\varphi - \psi\|_r$$

for those φ, $\varphi \in C_r$ with $\|\varphi\|_r \vee \|\psi\|_r \le M$.

(A2) For all φ, $\psi \in C_r$ and $k \in \mathbb{S}$, there exist $\alpha(k) \in \mathbb{R}$, $\beta(k) \in \mathbb{R}_+$ and $\rho \in \mathcal{P}_{2r}(\mathbb{R}_-)$ such that

$$2\langle \varphi(0) - \psi(0), b(\varphi, k) - b(\psi, k)\rangle \le \alpha(k)|\varphi(0) - \psi(0)|^2 + \beta(k) \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta).$$

Hypotheses

Assume $b : C_r \times \mathbb{S} \to \mathbb{R}^n$ and $\sigma : C_r \times \mathbb{S} \to \mathbb{R}^n \otimes \mathbb{R}^n$ satisfy the following conditions:

(A1) For any $k \in \mathbb{S}$ and any $M > 0$, there exists a constant $H_M > 0$ such that

$$|b(\varphi, k) - b(\psi, k)| \vee |\sigma(\varphi, k) - \sigma(\psi, k)| \leq H_M \|\varphi - \psi\|_r$$

for those φ, $\varphi \in C_r$ with $\|\varphi\|_r \vee \|\psi\|_r \leq M$.

(A2) For all φ, $\psi \in C_r$ and $k \in \mathbb{S}$, there exist $\alpha(k) \in \mathbb{R}$, $\beta(k) \in \mathbb{R}_+$ and $\rho \in \mathcal{P}_{2r}(\mathbb{R}_-)$ such that

$$\begin{aligned} &2\langle \varphi(0) - \psi(0), b(\varphi, k) - b(\psi, k)\rangle \\ &\quad \leq \alpha(k)|\varphi(0) - \psi(0)|^2 + \beta(k) \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta). \end{aligned}$$

Hypotheses (cont'd)

(A3) With ρ being determined in **(A2)**, for all $\varphi, \psi \in C_r$ and $k \in \mathbb{S}$, there exist $\gamma > 0$ such that

$$\|\sigma(\varphi, k) - \sigma(\psi, k)\|_{\mathrm{HS}}^2 \leq \gamma \int_{-\infty}^0 |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta).$$

In order to prove the existence and uniqueness for system (1) and (2), let $X^{(k)}(t)$, $k \in \mathbb{S}$, satisfy the following stochastic functional differential equation with infinite memory

$$\mathrm{d}X^{(k)}(t) = b(X_t^{(k)}, k)\mathrm{d}t + \sigma(X_t^{(k)}, k)\mathrm{d}W(t), \quad t \geq 0 \tag{3}$$

with initial data $X_0^{(k)} = \varphi^{(k)} \in C_r$.

Hypotheses (cont'd)

(A3) With ρ being determined in **(A2)**, for all $\varphi,\ \psi \in C_r$ and $k \in \mathbb{S}$, there exist $\gamma > 0$ such that

$$\|\sigma(\varphi, k) - \sigma(\psi, k)\|_{\mathsf{HS}}^2 \leq \gamma \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta).$$

In order to prove the existence and uniqueness for system (1) and (2), let $X^{(k)}(t)$, $k \in \mathbb{S}$, satisfy the following stochastic functional differential equation with infinite memory

$$\mathrm{d}X^{(k)}(t) = b(X_t^{(k)}, k)\mathrm{d}t + \sigma(X_t^{(k)}, k)\mathrm{d}W(t), \quad t \geq 0 \tag{3}$$

with initial data $X_0^{(k)} = \varphi^{(k)} \in C_r$.

Hypotheses (cont'd)

(A3) With ρ being determined in **(A2)**, for all φ, $\psi \in C_r$ and $k \in \mathbb{S}$, there exist $\gamma > 0$ such that

$$\|\sigma(\varphi, k) - \sigma(\psi, k)\|_{\mathsf{HS}}^2 \leq \gamma \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta).$$

In order to prove the existence and uniqueness for system (1) and (2), let $X^{(k)}(t)$, $k \in \mathbb{S}$, satisfy the following stochastic functional differential equation with infinite memory

$$\mathrm{d}X^{(k)}(t) = b(X_t^{(k)}, k)\mathrm{d}t + \sigma(X_t^{(k)}, k)\mathrm{d}W(t), \quad t \geq 0 \tag{3}$$

with initial data $X_0^{(k)} = \varphi^{(k)} \in C_r$.

Hypotheses (cont'd)

(A3) With ρ being determined in **(A2)**, for all $\varphi,\ \psi \in C_r$ and $k \in \mathbb{S}$, there exist $\gamma > 0$ such that

$$\|\sigma(\varphi, k) - \sigma(\psi, k)\|_{\mathrm{HS}}^2 \leq \gamma \int_{-\infty}^{0} |\varphi(\theta) - \psi(\theta)|^2 \rho(\mathrm{d}\theta).$$

In order to prove the existence and uniqueness for system (1) and (2), let $X^{(k)}(t)$, $k \in \mathbb{S}$, satisfy the following stochastic functional differential equation with infinite memory

$$\mathrm{d}X^{(k)}(t) = b(X_t^{(k)}, k)\mathrm{d}t + \sigma(X_t^{(k)}, k)\mathrm{d}W(t), \quad t \geq 0 \tag{3}$$

with initial data $X_0^{(k)} = \varphi^{(k)} \in C_r$.

Existence and Uniqueness

Under (**A1**)-(**A3**), from Wu, Yin and Mei (2017) we know that for each $k \in \mathbb{S}$, the equation (3) has a unique strong solution $X^{(k)}(t)$.

Note that the evolution of the discrete component Λ can be represented as a stochastic integral with respect to a Poisson random measure.

Then, by using a **successive approximation method**, we can prove the system (1) and (2) has a **unique strong solution** $(X(t), \Lambda(t))$ in addition to assuming that $Q(x) = (q_{kl}(x))$ is bounded.

The Markovian switching model is a important special case of the state-dependent switching model.

Existence and Uniqueness

Under (**A1**)-(**A3**), from Wu, Yin and Mei (2017) we know that for each $k \in \mathbb{S}$, the equation (3) has a unique strong solution $X^{(k)}(t)$.

Note that the evolution of the discrete component Λ can be represented as a stochastic integral with respect to a Poisson random measure.

Then, by using a **successive approximation method**, we can prove the system (1) and (2) has a **unique strong solution** $(X(t), \Lambda(t))$ in addition to assuming that $Q(x) = (q_{kl}(x))$ is bounded.

The Markovian switching model is a important special case of the state-dependent switching model.

Existence and Uniqueness

Under (**A1**)-(**A3**), from Wu, Yin and Mei (2017) we know that for each $k \in \mathbb{S}$, the equation (3) has a unique strong solution $X^{(k)}(t)$.

Note that the evolution of the discrete component Λ can be represented as a stochastic integral with respect to a Poisson random measure.

Then, by using a **successive approximation method**, we can prove the system (1) and (2) has a **unique strong solution** $(X(t), \Lambda(t))$ in addition to assuming that $Q(x) = (q_{kl}(x))$ is bounded.

The Markovian switching model is a important special case of the state-dependent switching model.

Existence and Uniqueness

Under (**A1**)-(**A3**), from Wu, Yin and Mei (2017) we know that for each $k \in \mathbb{S}$, the equation (3) has a unique strong solution $X^{(k)}(t)$.

Note that the evolution of the discrete component Λ can be represented as a stochastic integral with respect to a Poisson random measure.

Then, by using a **successive approximation method**, we can prove the system (1) and (2) has a **unique strong solution** $(X(t), \Lambda(t))$ in addition to assuming that $Q(x) = (q_{kl}(x))$ is bounded.

The Markovian switching model is a important special case of the state-dependent switching model.

Moment Boundedness

Moreover, we can prove that $X(t)$ satisfies the following property.

Lemma 2

Under assumptions $(\mathbf{A1})$-$(\mathbf{A3})$ *and that* $Q(x) = (q_{kl}(x))$ *is bounded, it holds that for any* $t > 0$,

$$\mathbb{E}\Big(\sup_{0<u\le t} e^{2ru}|X(u)|^2\Big) < \infty.$$

Write $X^{\varphi,k}(t)$ and $\Lambda^{\varphi,k}(t)$ to emphasize initial data $X_0 = \varphi$ and $\Lambda(0) = k$.

Moment Boundedness

Moreover, we can prove that $X(t)$ satisfies the following property.

Lemma 2

Under assumptions $(\mathbf{A1})$-$(\mathbf{A3})$ *and that* $Q(x) = (q_{kl}(x))$ *is bounded, it holds that for any* $t > 0$,

$$\mathbb{E}\Big(\sup_{0<u\le t} e^{2ru}|X(u)|^2\Big) < \infty.$$

Write $X^{\varphi,k}(t)$ and $\Lambda^{\varphi,k}(t)$ to emphasize initial data $X_0 = \varphi$ and $\Lambda(0) = k$.

Notaion and Hypotheses (cont'd)

In the sequel, we only consider the Markovian switching case. Put

$$\underline{\alpha} = \min_{k\in\mathbb{S}} \alpha(k), \quad \overline{\alpha} = \max_{k\in\mathbb{S}} \alpha(k), \quad \overline{\beta} = \max_{k\in\mathbb{S}} \beta(k), \tag{4}$$

$$\chi(k) = \gamma + \alpha(k) + (\gamma + \beta(k)) \int_{-\infty}^{0} e^{-(2r+\gamma-\underline{\alpha})\theta} \rho(\mathrm{d}\theta), \; k \in \mathbb{S}, \tag{5}$$

where $\alpha(k)$, $\beta(k)$, γ and ρ are introduced in (**A2**) and (**A3**). Moreover, set

$$\widehat{Q} := Q + \mathrm{diag}(\chi(1), \chi(2), \ldots, \chi(N)) \quad \text{and} \quad \eta := -\max_{\zeta\in\mathrm{spec}(\widehat{Q})} \mathrm{Re}(\zeta), \tag{6}$$

where $Q \equiv (q_{kl})$ is the generator of the Markov chain $\Lambda(t)$, and spec$(\widehat{Q})$ is the spectrum of $\widehat{Q}$.

(**A4**) Assume that $Q \equiv (q_{kl})$ is independent of x and irreducible, $\underline{\alpha} < 0$ and $\eta > 0$.

Notaion and Hypotheses (cont'd)

In the sequel, we only consider the Markovian switching case. Put

$$\underline{\alpha} = \min_{k\in\mathbb{S}} \alpha(k), \quad \overline{\alpha} = \max_{k\in\mathbb{S}} \alpha(k), \quad \overline{\beta} = \max_{k\in\mathbb{S}} \beta(k), \tag{4}$$

$$\chi(k) = \gamma + \alpha(k) + (\gamma + \beta(k)) \int_{-\infty}^{0} e^{-(2r+\gamma-\underline{\alpha})\theta} \rho(\mathrm{d}\theta), \; k \in \mathbb{S}, \tag{5}$$

where $\alpha(k)$, $\beta(k)$, γ and ρ are introduced in (**A2**) and (**A3**). Moreover, set

$$\widehat{Q} := Q + \mathrm{diag}(\chi(1), \chi(2), \dots, \chi(N)) \quad \text{and} \quad \eta := -\max_{\zeta\in\mathrm{spec}(\widehat{Q})} \mathrm{Re}(\zeta), \tag{6}$$

where $Q \equiv (q_{kl})$ is the generator of the Markov chain $\Lambda(t)$, and spec$(\widehat{Q})$ is the spectrum of $\widehat{Q}$.

(A4) Assume that $Q \equiv (q_{kl})$ is independent of x and irreducible, $\underline{\alpha} < 0$ and $\eta > 0$.

Notaion and Hypotheses (cont'd)

In the sequel, we only consider the Markovian switching case. Put

$$\underline{\alpha} = \min_{k\in\mathbb{S}} \alpha(k), \quad \overline{\alpha} = \max_{k\in\mathbb{S}} \alpha(k), \quad \overline{\beta} = \max_{k\in\mathbb{S}} \beta(k), \tag{4}$$

$$\chi(k) = \gamma + \alpha(k) + (\gamma + \beta(k)) \int_{-\infty}^{0} e^{-(2r+\gamma-\underline{\alpha})\theta} \rho(\mathrm{d}\theta), \ \ k \in \mathbb{S}, \tag{5}$$

where $\alpha(k)$, $\beta(k)$, γ and ρ are introduced in (**A2**) and (**A3**). Moreover, set

$$\widehat{Q} := Q + \mathrm{diag}(\chi(1), \chi(2), \ldots, \chi(N)) \ \ \text{and} \ \ \eta := -\max_{\zeta\in\mathrm{spec}(\widehat{Q})} \mathrm{Re}(\zeta), \tag{6}$$

where $Q \equiv \left(q_{kl}\right)$ is the generator of the Markov chain $\Lambda(t)$, and spec$(\widehat{Q})$ is the spectrum of $\widehat{Q}$.

(**A4**) Assume that $Q \equiv \left(q_{kl}\right)$ is independent of x and irreducible, $\underline{\alpha} < 0$ and $\eta > 0$.

Notaion and Hypotheses (cont'd)

In the sequel, we only consider the Markovian switching case. Put

$$\underline{\alpha} = \min_{k\in\mathbb{S}} \alpha(k), \quad \overline{\alpha} = \max_{k\in\mathbb{S}} \alpha(k), \quad \overline{\beta} = \max_{k\in\mathbb{S}} \beta(k), \tag{4}$$

$$\chi(k) = \gamma + \alpha(k) + (\gamma + \beta(k)) \int_{-\infty}^{0} e^{-(2r+\gamma-\underline{\alpha})\theta} \rho(\mathrm{d}\theta), \ \ k \in \mathbb{S}, \tag{5}$$

where $\alpha(k)$, $\beta(k)$, γ and ρ are introduced in (**A2**) and (**A3**). Moreover, set

$$\widehat{Q} := Q + \mathsf{diag}(\chi(1), \chi(2), \ldots, \chi(N)) \ \ \text{and} \ \ \eta := - \max_{\zeta \in \mathsf{spec}(\widehat{Q})} \mathsf{Re}(\zeta), \tag{6}$$

where $Q \equiv (q_{kl})$ is the generator of the Markov chain $\Lambda(t)$, and spec$(\widehat{Q})$ is the spectrum of $\widehat{Q}$.

(**A4**) Assume that $Q \equiv (q_{kl})$ is independent of x and irreducible, $\underline{\alpha} < 0$ and $\eta > 0$.

Useful Exponential Functional Estimates

From Proposition 4.1 in Bardet, Guérin and Malrieu (2010), we have the following useful lemma.

Lemma 3

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$, *there exist constants* $0 < c_1 < c_2 < \infty$ *such that for any* $i \in S$ *and* $0 \leq u < t$,

$$c_1 e^{-\eta(t-u)} \leq \mathbb{E}\Big[\exp\Big(\int_u^t \chi(\Lambda^i(v))\mathrm{d}v\Big)\Big] \leq c_2 e^{-\eta(t-u)},$$

where η *is defined in (6).*

For definiteness, we need to construct a appropriate probability space.

Useful Exponential Functional Estimates

From Proposition 4.1 in Bardet, Guérin and Malrieu (2010), we have the following useful lemma.

Lemma 3

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$, *there exist constants* $0 < c_1 < c_2 < \infty$ *such that for any* $i \in S$ *and* $0 \leq u < t$,

$$c_1 e^{-\eta(t-u)} \leq \mathbb{E}\Big[\exp\Big(\int_u^t \chi(\Lambda^i(v))\mathrm{d}v\Big)\Big] \leq c_2 e^{-\eta(t-u)},$$

where η *is defined in (6).*

For definiteness, we need to construct a appropriate probability space.

Useful Exponential Functional Estimates

From Proposition 4.1 in Bardet, Guérin and Malrieu (2010), we have the following useful lemma.

Lemma 3

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$, *there exist constants* $0 < c_1 < c_2 < \infty$ *such that for any* $i \in S$ *and* $0 \leq u < t$,

$$c_1 e^{-\eta(t-u)} \leq \mathbb{E}\Big[\exp\Big(\int_u^t \chi(\Lambda^i(v))\mathrm{d}v\Big)\Big] \leq c_2 e^{-\eta(t-u)},$$

where η *is defined in (6).*

For definiteness, we need to construct a appropriate probability space.

Construction of the Probability Space

Set

$$\Omega_1 = \{\omega \mid \omega : [0, \infty) \to \mathbb{R}^d \text{ is continuous with } \omega(0) = 0\}$$

endowed with the locally uniformly convergence topology and the Wiener measure $\mathbb{P}_1$ so that the coordinate process $W(t, \omega) = \omega(t)$, $t \geq 0$, is a standard d-dimensional Brownian motion.

Put

$$\Omega_2 = \{\omega \mid \omega : [0, \infty) \to \mathbb{S} \text{ is right continuous with left limit}\}$$

endowed with Skorokhod topology and a probability measure $\mathbb{P}_2$ so that the coordinate process $\Lambda(t, \omega) = \omega(t)$, $t \geq 0$, is a continuous time Markov chain with the generator $Q = (q_{kl})$.

Then, let

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2).$$

Construction of the Probability Space

Set

$$\Omega_1 = \{\omega \mid \omega : [0, \infty) \to \mathbb{R}^d \text{ is continuous with } \omega(0) = 0\}$$

endowed with the locally uniformly convergence topology and the Wiener measure $\mathbb{P}_1$ so that the coordinate process $W(t, \omega) = \omega(t)$, $t \geq 0$, is a standard d-dimensional Brownian motion.

Put

$$\Omega_2 = \{\omega \mid \omega : [0, \infty) \to \mathbb{S} \text{ is right continuous with left limit}\}$$

endowed with Skorokhod topology and a probability measure $\mathbb{P}_2$ so that the coordinate process $\Lambda(t, \omega) = \omega(t)$, $t \geq 0$, is a continuous time Markov chain with the generator $Q = (q_{kl})$.

Then, let

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2).$$

Construction of the Probability Space

Set

$$\Omega_1 = \{\omega \mid \omega : [0, \infty) \to \mathbb{R}^d \text{ is continuous with } \omega(0) = 0\}$$

endowed with the locally uniformly convergence topology and the Wiener measure $\mathbb{P}_1$ so that the coordinate process $W(t, \omega) = \omega(t)$, $t \geq 0$, is a standard d-dimensional Brownian motion.

Put

$$\Omega_2 = \{\omega \mid \omega : [0, \infty) \to \mathbb{S} \text{ is right continuous with left limit}\}$$

endowed with Skorokhod topology and a probability measure $\mathbb{P}_2$ so that the coordinate process $\Lambda(t, \omega) = \omega(t)$, $t \geq 0$, is a continuous time Markov chain with the generator $Q = (q_{kl})$.

Then, let

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathscr{B}(\Omega_1) \times \mathscr{B}(\Omega_2), \mathbb{P}_1 \times \mathbb{P}_2).$$

Construction of a Coupling Process

In order to study the convergence and exponential ergodicity in Wasserstein metric for the processes determined by the system (1) and (2).

We will construct a coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ for two copies $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ of the solution to the system (1) and (2).

To do so, we first construct the so-called basic coupling of the Markov chains $\Lambda(t)$ and $\Lambda'(t)$, then construct a combination of the independent coupling and basic coupling of $X(t)$ and $Y(t)$.

Note that we need to construct the coupling for $X(t)$ and $Y(t)$ by means of stochastic functional differential equations since they have no *good generators*.

Construction of a Coupling Process

In order to study the convergence and exponential ergodicity in Wasserstein metric for the processes determined by the system (1) and (2).

We will construct a coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ for two copies $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ of the solution to the system (1) and (2).

To do so, we first construct the so-called basic coupling of the Markov chains $\Lambda(t)$ and $\Lambda'(t)$, then construct a combination of the independent coupling and basic coupling of $X(t)$ and $Y(t)$.

Note that we need to construct the coupling for $X(t)$ and $Y(t)$ by means of stochastic functional differential equations since they have no *good generators*.

Construction of a Coupling Process

In order to study the convergence and exponential ergodicity in Wasserstein metric for the processes determined by the system (1) and (2).

We will construct a coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ for two copies $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ of the solution to the system (1) and (2).

To do so, we first construct the so-called basic coupling of the Markov chains $\Lambda(t)$ and $\Lambda'(t)$, then construct a combination of the independent coupling and basic coupling of $X(t)$ and $Y(t)$.

Note that we need to construct the coupling for $X(t)$ and $Y(t)$ by means of stochastic functional differential equations since they have no *good generators*.

Construction of a Coupling Process

In order to study the convergence and exponential ergodicity in Wasserstein metric for the processes determined by the system (1) and (2).

We will construct a coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ for two copies $(X(t), \Lambda(t))$ and $(Y(t), \Lambda'(t))$ of the solution to the system (1) and (2).

To do so, we first construct the so-called basic coupling of the Markov chains $\Lambda(t)$ and $\Lambda'(t)$, then construct a combination of the independent coupling and basic coupling of $X(t)$ and $Y(t)$.

Note that we need to construct the coupling for $X(t)$ and $Y(t)$ by means of stochastic functional differential equations since they have no *good generators*.

The Basic Coupling of $\Lambda(t)$ and $\Lambda'(t)$

Let the coupling process $\{(\Lambda(t), \Lambda'(t))\}$ be the Markov chain with phase space $\mathbb{S} \times \mathbb{S}$ and basic coupling operator (see Chen (2004))

$$\begin{aligned}\Omega f(k,l) = &\sum_m (q_{km} - q_{lm})^+ \big(f(m,l) - f(k,l)\big)\\ &+ \sum_m (q_{lm} - q_{km})^+ \big(f(k,m) - f(k,l)\big)\\ &+ \sum_m q_{km} \wedge q_{lm} \big(f(m,m) - f(k,l)\big),\end{aligned} \tag{7}$$

where f is a bounded function on $\mathbb{S} \times \mathbb{S}$.

Define the coupling time

$$S = \inf\{t \geq 0 : \Lambda(t) = \Lambda'(t)\},$$

then $\{\Lambda(t)\}$ and $\{\Lambda'(t)\}$ will move together from S onward.

The Basic Coupling of $\Lambda(t)$ and $\Lambda'(t)$

Let the coupling process $\{(\Lambda(t), \Lambda'(t))\}$ be the Markov chain with phase space $\mathbb{S} \times \mathbb{S}$ and basic coupling operator (see Chen (2004))

$$\begin{aligned} \Omega f(k,l) = & \sum_m (q_{km} - q_{lm})^+ \big(f(m,l) - f(k,l)\big) \\ & + \sum_m (q_{lm} - q_{km})^+ \big(f(k,m) - f(k,l)\big) \\ & + \sum_m q_{km} \wedge q_{lm} \big(f(m,m) - f(k,l)\big), \end{aligned} \tag{7}$$

where f is a bounded function on $\mathbb{S} \times \mathbb{S}$.

Define the coupling time

$$S = \inf\{t \geq 0 : \Lambda(t) = \Lambda'(t)\},$$

then $\{\Lambda(t)\}$ and $\{\Lambda'(t)\}$ will move together from S onward.

The Coupling $X(t)$ and $Y(t)$

For φ, $\psi \in C_r$ and k, $l \in \mathbb{S}$, set two $2d \times 2d$ matrices as follows:

$$\sigma(\varphi,\psi,k,l) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ 0 & \sigma(\psi,l) \end{pmatrix}, \ \sigma(\varphi,\psi,k) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix}.$$

Moreover, using the coupling time S of $\Lambda(t)$ and $\Lambda'(t)$, set

$$\sigma(t,\varphi,\psi,\Lambda(t),\Lambda'(t)) = \mathbf{1}_{[0,S)}(t)\sigma(\varphi,\psi,\Lambda(t),\Lambda'(t)) + \mathbf{1}_{[S,\infty)}(t)\sigma(\varphi,\psi,\Lambda(t)).$$

Let the coupling process $(X(t),Y(t))$ satisfy

$$\mathrm{d}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \sigma(t,X_t,Y_t,\Lambda(t),\Lambda'(t))\mathrm{d}\widetilde{W}(t) + \begin{pmatrix} b(X_t,\Lambda(t)) \\ b(Y_t,\Lambda'(t)) \end{pmatrix}\mathrm{d}t, \quad (8)$$

where $\widetilde{W}(t)$ is a $2d$-dimensional Brownian motion independent of $(\Lambda(t),\Lambda'(t))$.

The Coupling $X(t)$ and $Y(t)$

For φ, $\psi \in C_r$ and k, $l \in \mathbb{S}$, set two $2d \times 2d$ matrices as follows:

$$\sigma(\varphi,\psi,k,l) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ 0 & \sigma(\psi,l) \end{pmatrix}, \ \sigma(\varphi,\psi,k) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix}.$$

Moreover, using the coupling time S of $\Lambda(t)$ and $\Lambda'(t)$, set

$$\sigma(t,\varphi,\psi,\Lambda(t),\Lambda'(t)) = \mathbf{1}_{[0,S)}(t)\sigma(\varphi,\psi,\Lambda(t),\Lambda'(t)) + \mathbf{1}_{[S,\infty)}(t)\sigma(\varphi,\psi,\Lambda(t)).$$

Let the coupling process $(X(t),Y(t))$ satisfy

$$\mathrm{d}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \sigma(t,X_t,Y_t,\Lambda(t),\Lambda'(t))\mathrm{d}\widetilde{W}(t) + \begin{pmatrix} b(X_t,\Lambda(t)) \\ b(Y_t,\Lambda'(t)) \end{pmatrix}\mathrm{d}t, \quad (8)$$

where $\widetilde{W}(t)$ is a $2d$-dimensional Brownian motion independent of $(\Lambda(t),\Lambda'(t))$.

The Coupling $X(t)$ and $Y(t)$

For φ, $\psi \in C_r$ and k, $l \in \mathbb{S}$, set two $2d \times 2d$ matrices as follows:

$$\sigma(\varphi,\psi,k,l) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ 0 & \sigma(\psi,l) \end{pmatrix}, \ \sigma(\varphi,\psi,k) = \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix}.$$

Moreover, using the coupling time S of $\Lambda(t)$ and $\Lambda'(t)$, set

$$\sigma(t,\varphi,\psi,\Lambda(t),\Lambda'(t)) = \mathbf{1}_{[0,S)}(t)\sigma(\varphi,\psi,\Lambda(t),\Lambda'(t)) + \mathbf{1}_{[S,\infty)}(t)\sigma(\varphi,\psi,\Lambda(t)).$$

Let the coupling process $(X(t),Y(t))$ satisfy

$$\mathrm{d}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \sigma\big(t,X_t,Y_t,\Lambda(t),\Lambda'(t)\big)\mathrm{d}\widetilde{W}(t) + \begin{pmatrix} b\big(X_t,\Lambda(t)\big) \\ b\big(Y_t,\Lambda'(t)\big) \end{pmatrix}\mathrm{d}t, \qquad (8)$$

where $\widetilde{W}(t)$ is a $2d$-dimensional Brownian motion independent of $(\Lambda(t),\Lambda'(t))$.

Explanation on the Coupling $X(t)$ and $Y(t)$

Since

$$\sigma(\varphi,\psi,k,l)\sigma(\varphi,\psi,k,l)^* = \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & 0 \\ 0 & \sigma(\psi,l)\sigma(\psi,l)^* \end{pmatrix}$$

and

$$\begin{aligned}\sigma(\varphi,\psi,k)\sigma(\varphi,\psi,k)^* &= \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix}\begin{pmatrix} \sigma(\varphi,k)^* & \sigma(\psi,k)^* \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & \sigma(\varphi,k)\sigma(\psi,k)^* \\ \sigma(\psi,k)\sigma(\varphi,k)^* & \sigma(\psi,k)\sigma(\psi,k)^* \end{pmatrix},\end{aligned}$$

so $(X(t),Y(t))$ determined by equation (8) is the independent coupling on $[0,S)$ and the basic coupling on $[S,\infty)$ of $X(t)$ and $Y(t)$, where S is the coupling time of $\Lambda(t)$ and $\Lambda'(t)$.

Namely, before $\Lambda(t)$ and $\Lambda'(t)$ are coupled together, $X(t)$ and $Y(t)$ run independently, whereas from S onward, $X(t)$ and $Y(t)$ couple each other in the basic coupling manner.

Explanation on the Coupling $X(t)$ and $Y(t)$

Since

$$\sigma(\varphi,\psi,k,l)\sigma(\varphi,\psi,k,l)^* = \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & 0 \\ 0 & \sigma(\psi,l)\sigma(\psi,l)^* \end{pmatrix}$$

and

$$\begin{aligned} \sigma(\varphi,\psi,k)\sigma(\varphi,\psi,k)^* &= \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix} \begin{pmatrix} \sigma(\varphi,k)^* & \sigma(\psi,k)^* \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & \sigma(\varphi,k)\sigma(\psi,k)^* \\ \sigma(\psi,k)\sigma(\varphi,k)^* & \sigma(\psi,k)\sigma(\psi,k)^* \end{pmatrix}, \end{aligned}$$

so $(X(t),Y(t))$ determined by equation (8) is the independent coupling on $[0,S)$ and the basic coupling on $[S,\infty)$ of $X(t)$ and $Y(t)$, where S is the coupling time of $\Lambda(t)$ and $\Lambda'(t)$.

Namely, before $\Lambda(t)$ and $\Lambda'(t)$ are coupled together, $X(t)$ and $Y(t)$ run independently, whereas from S onward, $X(t)$ and $Y(t)$ couple each other in the basic coupling manner.

Explanation on the Coupling $X(t)$ and $Y(t)$

Since

$$\sigma(\varphi,\psi,k,l)\sigma(\varphi,\psi,k,l)^* = \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & 0 \\ 0 & \sigma(\psi,l)\sigma(\psi,l)^* \end{pmatrix}$$

and

$$\begin{aligned}\sigma(\varphi,\psi,k)\sigma(\varphi,\psi,k)^* &= \begin{pmatrix} \sigma(\varphi,k) & 0 \\ \sigma(\psi,k) & 0 \end{pmatrix}\begin{pmatrix} \sigma(\varphi,k)^* & \sigma(\psi,k)^* \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma(\varphi,k)\sigma(\varphi,k)^* & \sigma(\varphi,k)\sigma(\psi,k)^* \\ \sigma(\psi,k)\sigma(\varphi,k)^* & \sigma(\psi,k)\sigma(\psi,k)^* \end{pmatrix},\end{aligned}$$

so $(X(t),Y(t))$ determined by equation (8) is the independent coupling on $[0,S)$ and the basic coupling on $[S,\infty)$ of $X(t)$ and $Y(t)$, where S is the coupling time of $\Lambda(t)$ and $\Lambda'(t)$.

Namely, before $\Lambda(t)$ and $\Lambda'(t)$ are coupled together, $X(t)$ and $Y(t)$ run independently, whereas from S onward, $X(t)$ and $Y(t)$ couple each other in the basic coupling manner.

The Coupling $(X(t), \Lambda(t)$ and Itself

For the coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ determined by system (7) and (8), we have the following observation.

When $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ starts from (φ, ψ, k, k), equation (8) can be rewritten as

$$\begin{cases} X^{\varphi,k}(t) = \varphi(0) + \int_0^t \sigma(X_u^{\varphi,k}, \Lambda^k(u))\mathrm{d}W(u) + \int_0^t b(X_u^{\varphi,k}, \Lambda^k(u))\mathrm{d}u, \\ Y^{\psi,k}(t) = \psi(0) + \int_0^t \sigma(Y_u^{\psi,k}, \Lambda^k(u))\mathrm{d}W(u) + \int_0^t b(Y_u^{\psi,k}, \Lambda^k(u))\mathrm{d}u, \end{cases} \tag{9}$$

where $W(t)$ is a d-dimensional Brownian motion.

For convenience, we will write the above $Y^{\psi,k}(t)$ as $X^{\psi,k}(t)$ in the sequel.

The Coupling $(X(t), \Lambda(t)$ and Itself

For the coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ determined by system (7) and (8), we have the following observation.

When $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ starts from (φ, ψ, k, k), equation (8) can be rewritten as

$$\begin{cases} X^{\varphi,k}(t) = \varphi(0) + \int_0^t \sigma(X_u^{\varphi,k}, \Lambda^k(u)) \mathrm{d}W(u) + \int_0^t b(X_u^{\varphi,k}, \Lambda^k(u)) \mathrm{d}u, \\ Y^{\psi,k}(t) = \psi(0) + \int_0^t \sigma(Y_u^{\psi,k}, \Lambda^k(u)) \mathrm{d}W(u) + \int_0^t b(Y_u^{\psi,k}, \Lambda^k(u)) \mathrm{d}u, \end{cases} \quad (9)$$

where $W(t)$ is a d-dimensional Brownian motion.

For convenience, we will write the above $Y^{\psi,k}(t)$ as $X^{\psi,k}(t)$ in the sequel.

The Coupling $(X(t), \Lambda(t)$ and Itself

For the coupling process $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ determined by system (7) and (8), we have the following observation.

When $(X(t), Y(t), \Lambda(t), \Lambda'(t))$ starts from (φ, ψ, k, k), equation (8) can be rewritten as

$$\begin{cases} X^{\varphi,k}(t) = \varphi(0) + \int_0^t \sigma(X_u^{\varphi,k}, \Lambda^k(u)) \mathrm{d}W(u) + \int_0^t b(X_u^{\varphi,k}, \Lambda^k(u)) \mathrm{d}u, \\ Y^{\psi,k}(t) = \psi(0) + \int_0^t \sigma(Y_u^{\psi,k}, \Lambda^k(u)) \mathrm{d}W(u) + \int_0^t b(Y_u^{\psi,k}, \Lambda^k(u)) \mathrm{d}u, \end{cases} \tag{9}$$

where $W(t)$ is a d-dimensional Brownian motion.

For convenience, we will write the above $Y^{\psi,k}(t)$ as $X^{\psi,k}(t)$ in the sequel.

Convergence and Boundedness of $X(t)$

By virtue of the above coupling, we can prove the following convergence and boundedness results for $X(t)$.

Theorem 4

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}|X^{\varphi,k}(t) - X^{\psi,k}(t)|^2 \leq Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 5

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for each* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t\geq 0} \mathbb{E}|X^{\varphi,k}(t)|^2 \leq C(1 + \|\varphi\|_r^2).$$

Convergence and Boundedness of $X(t)$

By virtue of the above coupling, we can prove the following convergence and boundedness results for $X(t)$.

Theorem 4

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}|X^{\varphi,k}(t) - X^{\psi,k}(t)|^2 \le Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 5

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for each* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t\ge 0} \mathbb{E}|X^{\varphi,k}(t)|^2 \le C(1 + \|\varphi\|_r^2).$$

Convergence and Boundedness of $X(t)$

By virtue of the above coupling, we can prove the following convergence and boundedness results for $X(t)$.

Theorem 4

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}|X^{\varphi,k}(t) - X^{\psi,k}(t)|^2 \leq Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 5

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for each* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t\geq 0} \mathbb{E}|X^{\varphi,k}(t)|^2 \leq C(1 + \|\varphi\|_r^2).$$

Convergence of Boundedness X_t

Furthermore, by virtue of the above coupling, the two previous theorems and Lemma 2, we can prove the following convergence and boundedness results for the **segment process** X_t.

Theorem 6

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}\|X_t^{\varphi,k} - X_t^{\psi,k}\|_r^2 \leq Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 7

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for all* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t \geq 0} \mathbb{E}\|X_t^{\varphi,k}\|_r^2 \leq C(1 + \|\psi\|_r^2).$$

Convergence of Boundedness X_t

Furthermore, by virtue of the above coupling, the two previous theorems and Lemma 2, we can prove the following convergence and boundedness results for the **segment process** X_t.

Theorem 6

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}\|X_t^{\varphi,k} - X_t^{\psi,k}\|_r^2 \le Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 7

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for all* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t \ge 0} \mathbb{E}\|X_t^{\varphi,k}\|_r^2 \le C(1 + \|\psi\|_r^2).$$

Convergence of Boundedness X_t

Furthermore, by virtue of the above coupling, the two previous theorems and Lemma 2, we can prove the following convergence and boundedness results for the **segment process** X_t.

Theorem 6

Suppose $(\mathbf{A1})$–$(\mathbf{A4})$ *hold. Then there exist constants* C, $\lambda > 0$ *such that for each* φ, $\psi \in C_r$, $k \in \mathbb{S}$, *and any* $t > 0$,

$$\mathbb{E}\|X_t^{\varphi,k} - X_t^{\psi,k}\|_r^2 \leq Ce^{-\lambda t}\|\varphi - \psi\|_r^2.$$

Theorem 7

Suppose $(\mathbf{A1})$-$(\mathbf{A4})$ *hold. Then there exists constant* $C > 0$ *such that for all* $\varphi \in C_r$ *and* $k \in \mathbb{S}$,

$$\sup_{t\geq 0} \mathbb{E}\|X_t^{\varphi,k}\|_r^2 \leq C(1 + \|\psi\|_r^2).$$

Wasserstein Metric

Set $E = C_r \times \mathbb{S}$. define a distance d on E:

$$d((\varphi,k),(\psi,l)) := \|\varphi-\psi\|_r + \ell(k,l), \quad (\varphi,k),(\psi,l) \in E,$$

where ℓ is the discrete distance on S. Thus, $(E, d(\cdot,\cdot))$ is a polish space.

Then, as in Chen (2004) we can define the Wasserstein metric between two probability measures μ, $\nu \in \mathcal{P}(E)$ as follows:

$$\mathcal{W}(\mu,\nu) = \inf_{\varrho \in \mathscr{C}(\mu,\nu)} \int_{E\times E} d((\varphi,k),(\psi,l))\varrho(\mathrm{d}\varphi \times \mathrm{d}\{k\}, \mathrm{d}\psi \times \mathrm{d}\{l\}),$$

where $\mathscr{C}(\mu,\nu)$ is the set of all coupling probability measures on $E \times E$ with marginals μ and ν.

Wasserstein Metric

Set $E = C_r \times \mathbb{S}$. define a distance d on E:

$$d((\varphi,k),(\psi,l)) := \|\varphi-\psi\|_r + \ell(k,l), \quad (\varphi,k),(\psi,l)\in E,$$

where ℓ is the discrete distance on S. Thus, $(E, d(\cdot,\cdot))$ is a polish space.

Then, as in Chen (2004) we can define the Wasserstein metric between two probability measures μ, $\nu \in \mathcal{P}(E)$ as follows:

$$\mathcal{W}(\mu,\nu) = \inf_{\varrho\in\mathscr{C}(\mu,\nu)} \int_{E\times E} d((\varphi,k),(\psi,l))\varrho(\mathrm{d}\varphi\times\mathrm{d}\{k\},\mathrm{d}\psi\times\mathrm{d}\{l\}),$$

where $\mathscr{C}(\mu,\nu)$ is the set of all coupling probability measures on $E\times E$ with marginals μ and ν.

Wasserstein Metric

Set $E = C_r \times \mathbb{S}$. define a distance d on E:

$$d((\varphi, k), (\psi, l)) := \|\varphi - \psi\|_r + \ell(k, l), \quad (\varphi, k), (\psi, l) \in E,$$

where ℓ is the discrete distance on S. Thus, $(E, d(\cdot, \cdot))$ is a polish space.

Then, as in Chen (2004) we can define the Wasserstein metric between two probability measures μ, $\nu \in \mathcal{P}(E)$ as follows:

$$\mathcal{W}(\mu, \nu) = \inf_{\varrho \in \mathscr{C}(\mu,\nu)} \int_{E \times E} d((\varphi, k), (\psi, l)) \varrho(\mathrm{d}\varphi \times \mathrm{d}\{k\}, \mathrm{d}\psi \times \mathrm{d}\{l\}),$$

where $\mathscr{C}(\mu, \nu)$ is the set of all coupling probability measures on $E \times E$ with marginals μ and ν.

Wasserstein Metric

Set $E = C_r \times \mathbb{S}$. define a distance d on E:

$$d((\varphi,k),(\psi,l)) := \|\varphi-\psi\|_r + \ell(k,l), \quad (\varphi,k),(\psi,l)\in E,$$

where ℓ is the discrete distance on S. Thus, $(E, d(\cdot,\cdot))$ is a polish space.

Then, as in Chen (2004) we can define the Wasserstein metric between two probability measures μ, $\nu\in\mathcal{P}(E)$ as follows:

$$\mathcal{W}(\mu,\nu) = \inf_{\varrho\in\mathscr{C}(\mu,\nu)} \int_{E\times E} d((\varphi,k),(\psi,l))\varrho(\mathrm{d}\varphi\times\mathrm{d}\{k\},\mathrm{d}\psi\times\mathrm{d}\{l\}),$$

where $\mathscr{C}(\mu,\nu)$ is the set of all coupling probability measures on $E\times E$ with marginals μ and ν.

Feller Property of $(X_t, \Lambda(t))$

We now consider the Markov process $(X_t^{\varphi,k}, \Lambda^k(t))$ on the Polish space (E, d). Let $P_t((\phi, k), A)$ denote its transition probability. For the existence of invariant measure, we need to prove the Feller property for $(X_t, \Lambda(t))$.

Proposition 8

Under $(\mathbf{A1})$-$(\mathbf{A3})$*, the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *has the Feller property.*

For a later use, put

$$\mathcal{P}_1(E) := \Big\{\mu \in \mathcal{P}(E); \int_E d((\varphi, k), (\varphi_1, k_1)\mu(\mathrm{d}\varphi \times \mathrm{d}\{k\}) < \infty\Big\},$$

where $(\varphi_1, k_1) \in E$ is arbitrarily given. This space does not depend on the choice of the point (φ_1, k_1).

Feller Property of $(X_t, \Lambda(t))$

We now consider the Markov process $(X_t^{\varphi,k}, \Lambda^k(t))$ on the Polish space (E, d). Let $P_t((\phi, k), A)$ denote its transition probability. For the existence of invariant measure, we need to prove the Feller property for $(X_t, \Lambda(t))$.

Proposition 8

Under $(\mathbf{A1})$-$(\mathbf{A3})$, *the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *has the Feller property.*

For a later use, put

$$\mathcal{P}_1(E) := \Big\{\mu \in \mathcal{P}(E); \int_E d((\varphi, k), (\varphi_1, k_1)\mu(\mathrm{d}\varphi \times \mathrm{d}\{k\}) < \infty\Big\},$$

where $(\varphi_1, k_1) \in E$ is arbitrarily given. This space does not depend on the choice of the point (φ_1, k_1).

Feller Property of $(X_t, \Lambda(t))$

We now consider the Markov process $(X_t^{\varphi,k}, \Lambda^k(t))$ on the Polish space (E, d). Let $P_t((\phi, k), A)$ denote its transition probability. For the existence of invariant measure, we need to prove the Feller property for $(X_t, \Lambda(t))$.

Proposition 8

Under $(\mathbf{A1})$-$(\mathbf{A3})$, *the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *has the Feller property.*

For a later use, put

$$\mathcal{P}_1(E) := \left\{\mu \in \mathcal{P}(E); \int_E d((\varphi, k), (\varphi_1, k_1)\mu(\mathrm{d}\varphi \times \mathrm{d}\{k\}) < \infty\right\},$$

where $(\varphi_1, k_1) \in E$ is arbitrarily given. This space does not depend on the choice of the point (φ_1, k_1).

Exponential Ergodicity of $(X_t, \Lambda(t))$

Finally, by virtue of estimating the coupling time S, and using the coupling constructed by system system (7) and (8), Theorems 6 and 7, we can prove the exponential ergodicity for $(X_t^{\varphi,k}, \Lambda^k(t))$.

Theorem 9

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$, *the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *admits a unique invariant measure* $\pi \in \mathcal{P}_1(E)$ *and the transition probability* $P_t((\varphi, k), \cdot)$ *converges to it exponentially in the Wasserstein metric. That is, there exist constants* C, $\kappa > 0$ *such that for each* $(\varphi, k) \in E$,

$$\mathcal{W}(P_t((\varphi, k), \cdot), \pi) \leq C\Big(1 + \|\varphi\|_r + \int_E \|\psi\|_r \pi(\mathrm{d}\psi \times \mathrm{d}\{l\})\Big)e^{-\kappa t}.$$

Exponential Ergodicity of $(X_t, \Lambda(t))$

Finally, by virtue of estimating the coupling time S, and using the coupling constructed by system system (7) and (8), Theorems 6 and 7, we can prove the exponential ergodicity for $(X_t^{\varphi,k}, \Lambda^k(t))$.

Theorem 9

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$, *the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *admits a unique invariant measure* $\pi \in \mathcal{P}_1(E)$ *and the transition probability* $P_t((\varphi,k),\cdot)$ *converges to it exponentially in the Wasserstein metric. That is, there exist constants* C, $\kappa > 0$ *such that for each* $(\varphi, k) \in E$,

$$\mathcal{W}(P_t((\varphi,k),\cdot),\pi) \leq C\Big(1 + \|\varphi\|_r + \int_E \|\psi\|_r \pi(\mathrm{d}\psi \times \mathrm{d}\{l\})\Big) e^{-\kappa t}.$$

Exponential Ergodicity of $(X_t, \Lambda(t))$

Finally, by virtue of estimating the coupling time S, and using the coupling constructed by system system (7) and (8), Theorems 6 and 7, we can prove the exponential ergodicity for $(X_t^{\varphi,k}, \Lambda^k(t))$.

Theorem 9

Under assumptions $(\mathbf{A1})$-$(\mathbf{A4})$*, the process* $(X_t^{\varphi,k}, \Lambda^k(t))_{t\geq 0}$ *admits a unique invariant measure* $\pi \in \mathcal{P}_1(E)$ *and the transition probability* $P_t((\varphi,k),\cdot)$ *converges to it exponentially in the Wasserstein metric. That is, there exist constants* C, $\kappa > 0$ *such that for each* $(\varphi, k) \in E$,

$$\mathcal{W}(P_t((\varphi,k),\cdot),\pi) \leq C\Big(1 + \|\varphi\|_r + \int_E \|\psi\|_r \pi(\mathrm{d}\psi \times \mathrm{d}\{l\})\Big)e^{-\kappa t}.$$

Some References

1. Bao J., Shao J., Yuan C., *Potential Anal.*, 44(2016), 707–727.
2. Bao J, Wang F.Y, Yuan C., *Math. Nachr.*, 293(2020), 1675–1690.
3. Bardet J., Guérin H., Malrieu F., *ALEA Lat. Am. J. Probab. Math. Stat.*, 7(2010), 151–170.
4. Chen M.F., From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore, 2004.
5. Chen M.F., Li S.F., *Ann. Probab.*, 17(1989), 151–177.
6. Li J., Xi F., *Front. Math. China*, 10(2021), 499–523.
7. Mohammed S-E A., Stochastic Functional Differential Equations. Harlow-New York: Longman, 1986.
8. Shi B., Wang Y., Wu F., *SIAM J. Contr. Optim.*, 60(2022), 2658–2683.
9. Wu F, Yin G, Mei H., *J. Differential. Equations*, 262(2017), 1226–1252.
10. Xi F., *Stat. Probab. Letters*, 68(2004), 273–286.
11. Xi F., *Stoch. Process. Appl.*, 119(2009), 2198–2221.
12. Xi F, Yin G, *Sci. China Math.*, 54(2011), 2651–2667.

Thank You Very Much!